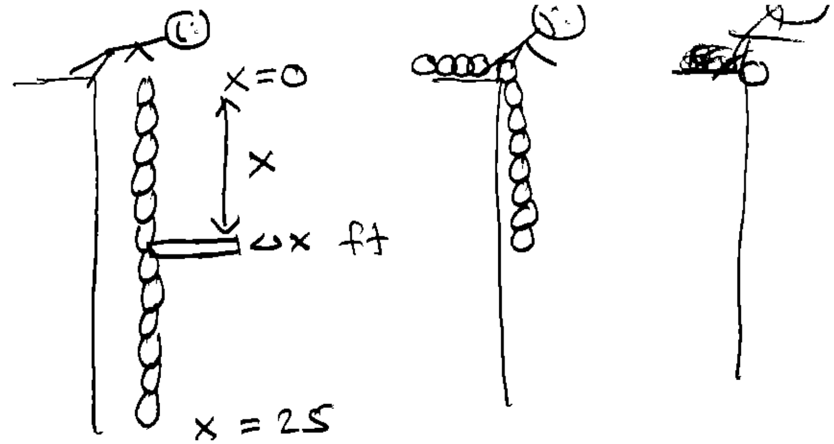
Examples:

1. (Chains/Cables) You are lifting a heavy chain to the top of a building. The chain has a density of 3 lbs/foot. The chain hangs over the side by 25 feet before you start pulling it up. How much work is done in pulling the chain all the way to the top?



WEIGHT OF SUBDIVISION $= 3 \frac{\text{lbs}}{\cancel{\text{ft}}} \cdot \Delta x \cancel{\text{ft}} = 3\Delta x$

DIST $= x$

$$\text{WORK} = \int_0^{25} \overbrace{x}^{\text{DIST}} \; \overbrace{3\,dx}^{\text{FORCE}}$$

$$= \frac{1}{2}x^2 \cdot 3 \Big|_0^{25}$$

$$= \frac{3}{2}\left((25)^2 - (0)^2\right)$$

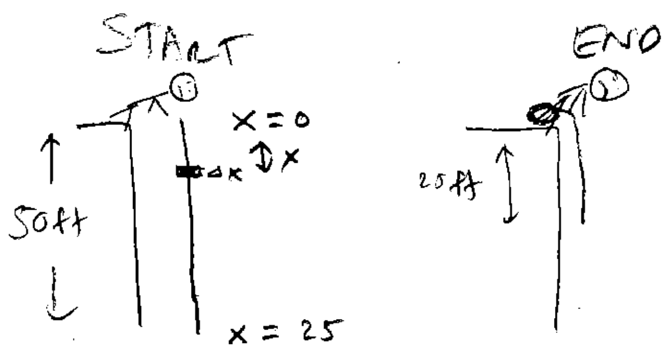
$$= \frac{3}{2} \cdot 625$$

$$= \boxed{937.5 \text{ ft-lbs}}$$

Example:

A 50 foot cable with density 4 lbs/ft is hanging over the side of a tall building.

Find the total work done in lifting the cable half way up.



For $0 \le x \le 25$:

FORCE = $4\Delta x$

DIST = x

$$\int_0^{25} x \, 4\,dx = 2x^2\Big|_0^{25}$$

$$= 1250 \text{ ft-lbs}$$

For $25 \le x \le 50$:

FORCE = $4\Delta x$

DIST = 25

$$\int_{25}^{50} 25 \cdot 4\,dx = 100x\Big|_{25}^{50}$$

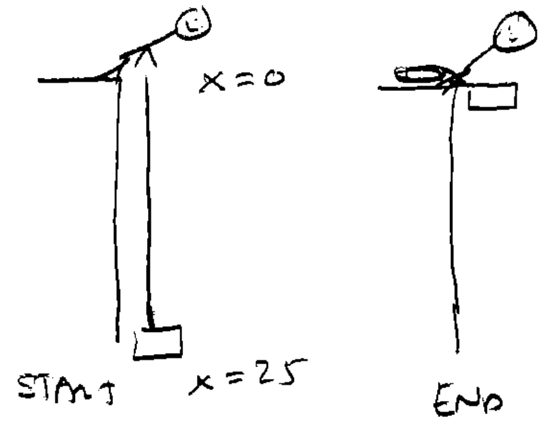
$$= 100(50-25)$$

$$= 2500 \text{ ft-lbs}$$

TOTAL = $1250 + 2500 = \boxed{3750 \text{ ft-lbs}}$

Example: (*You do – like HW*) 3

A cable with density ~~4~~ lbs/ft is being used to lift a 50 pound weight from the ground to the top of a 25 foot building. Find the total work done.



Step 1: Draw a picture.

Step 2: Break up the problem:

(a) Work to lift the 50 lbs weight?

(b) Work to lift the cable?

Step 3: Add these together.

OBJECT

$F = 50$ lbs

$D = \Delta x$

$\int_0^{25} 50\,dx$

$50x \Big|_0^{25}$

$50 \cdot (25 - 0)$

1250 ft-lbs

CABLE

$F = 3\Delta x$

$D = x$

$\int_0^{25} x\, 3\,dx$

$\frac{3}{2}x^2 \Big|_0^{25}$

$\frac{3}{2}((25)^2 - 0)$

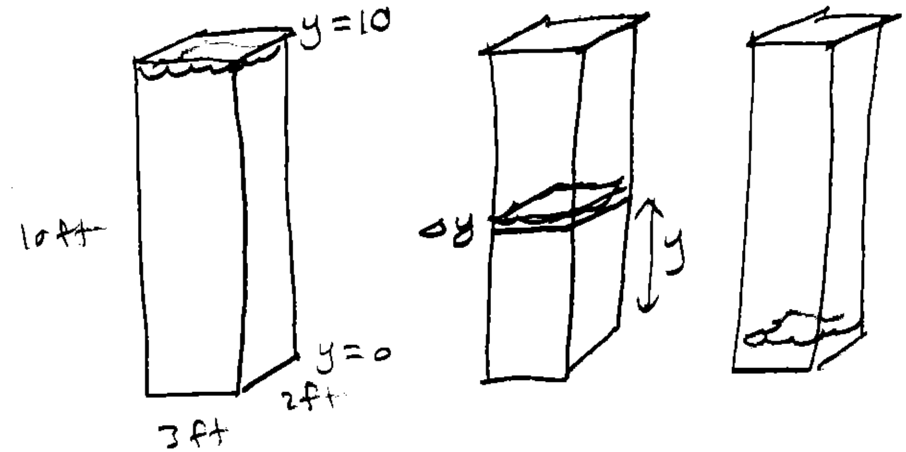
937.5 ft-lbs

TOTAL

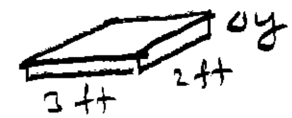
$$\int_0^{25} 50 + 3x\,dx = 1250 + 937.5$$

$$= 2187.5 \text{ ft-lbs}$$

2. (Pumping Liquid) You are pumping water out of a tank. The tank is a rectangular box with a base of 2 ft by 3 ft and height of 10ft. The density of water is 62.5 lbs/ft^3.



If the tank starts full, how much work is done in pumping all the water to the top and out over the side?



$$\text{WORK} = \int_0^{10} \overbrace{(10-y)}^{\text{DIST}}\ \overbrace{\underbrace{62.5}_{\frac{\text{lbs}}{\text{ft}^3}} \cdot \underbrace{6\,dy}_{\text{ft}^3}}^{\text{FORCE}}$$

$$\text{WEIGHT} = 62.5 \frac{\text{lbs}}{\text{ft}^3} \cdot 3 \cdot 2 \cdot \Delta y \ \text{ft}^3$$

$$= 62.5 \cdot 6\,\Delta y$$

$$\text{DIST LIFTED} = 10 - y$$

$$= 375 \int_0^{10} 10 - y \, dy$$

$$= 375 \left(10y - \tfrac{1}{2}y^2 \Big|_0^{10}\right)$$

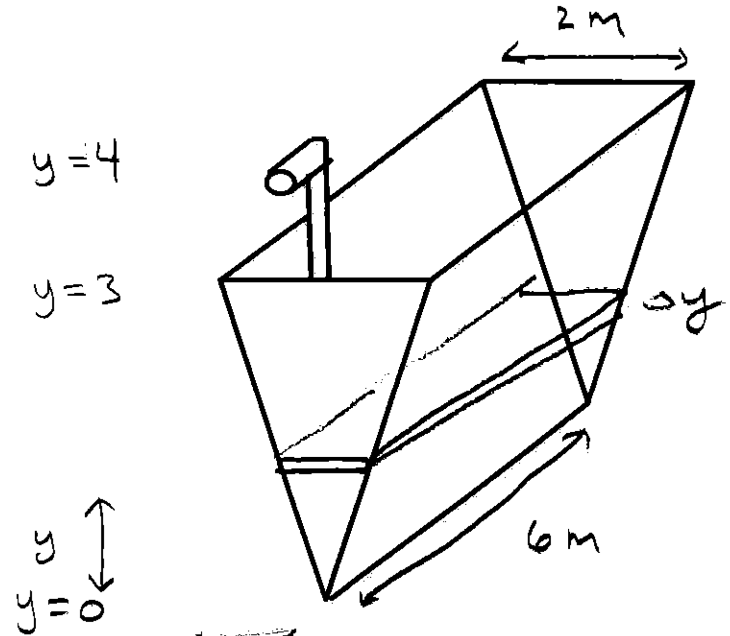
$$= 375\,(100 - 50) = 375 \cdot 50 = \boxed{18{,}750 \text{ ft-lbs}}$$

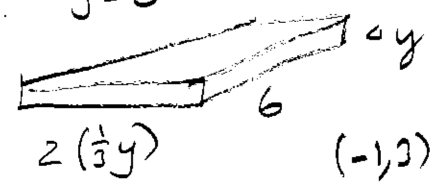
Example:

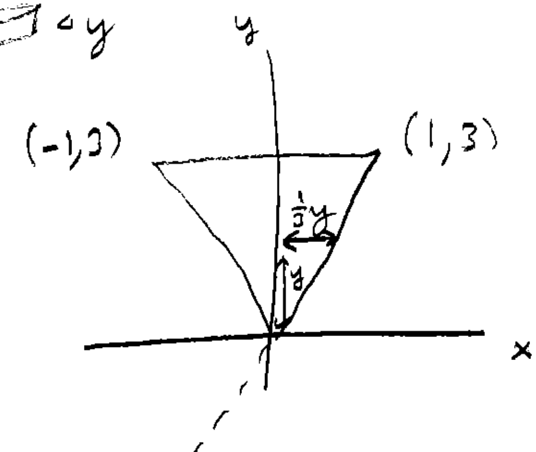
Consider the tank show at right.

The height is 3 meters, the width at the top is 2 meters and the length is 6 meters. Also we are pumping the water up to 4 meters above the ground (1 meter above the top edge).

If it starts full, how much work is done to pump it all out?







WEIGHT OF HORIZ. SLICE $= 9800 \frac{N}{m^3} \cdot \frac{2}{3}y \cdot 6\Delta y \ m^3$

$= 9800 \cdot 4y\Delta y$

DIST LIFTED $= 4 - y$

$$\int_0^3 \overbrace{(4-y)}^{\text{DIST}} \overbrace{\underset{\frac{N}{m^3}}{9800} \cdot \underbrace{4 \cdot y\, dy}_{m^3}}^{\text{FORCE}}$$

slope $= \frac{3-0}{1-0} = 3$

$y = 3x$

$x = \frac{1}{3}y$

$$= 39200 \int_0^3 (4-y)y\,dy = 39200 \int_0^3 4y - y^2\,dy$$

$$= 39200\left(2y^2 - \frac{1}{3}y^3 \Big|_0^3\right) = 39200((18-9)-0) = \boxed{352{,}800 \text{ Joules}}$$

Quick Summary:

$$\text{Work} = \lim_{n\to\infty}\sum_{i=1}^{n}(FORCE)(DIST)$$

$$= \int_a^b (FORCE)(DIST)$$

Problem type 1: (Leaky bucket/spring)

Leaking at constant rate → f(x) = mx+b

Spring (Hooke's Law) → f(x) = kx

Force given → f(x) = force

FORCE = $f(x_i)$, DISTANCE = Δx

WORK = $\int_a^b f(x)dx$

Problem type 2: (Chain/pumping)

FORCE = weight of a horizontal slice

DIST = distance moved by that slice

Chain:

k = density = force per distance

FORCE = weight of slice = kΔx

DIST = distance moved by slice

(typically *x* if you label like me)

WORK = $\int_0^b x\, kdx$

Pumping:

Density of water

= 1000 kg/m^3 = 9800 N/m^3

= 62.5 lbs/ft^3

k = density = weight per volume

FORCE = k vol = k(hor. slice area)Δy

DIST = distance moved by slice

(typically *a-y* if you label like me)

WORK = $\int_0^b (a-y)k(slice\ area)dy$

6.5 Average Value

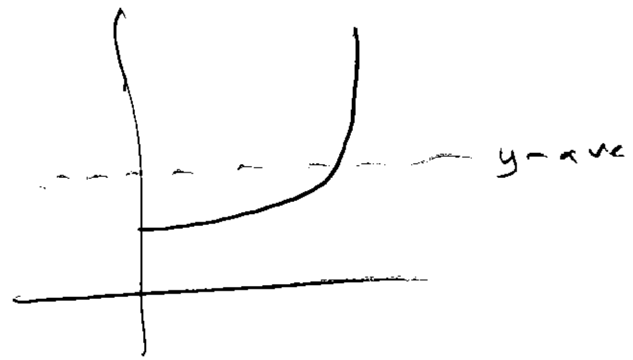
The average value of the *n* numbers:

$$y_1, y_2, y_3, \ldots, y_n$$

is given by

$$\frac{y_1+y_2+y_3+\cdots+y_n}{n} = y_1\frac{1}{n} + \cdots + y_n\frac{1}{n}.$$

Goal: We want the average value of **all** the y-values of some function y = f(x) over an interval x = a to x = b.



Ex] 4 TEST SCORE

60, 70, 80, 90

$$\text{AVE TEST SCORE} = \frac{60+70+80+90}{4} = 75$$

Derivation:

1. Break into n equal subdivisions

$$\Delta x = \frac{b-a}{n}, \text{ which means } \frac{\Delta x}{b-a} = \frac{1}{n}$$

2. Compute y-value at each tick mark

$$y_1 = f(x_1), y_2 = f(x_2), \dots, y_n = f(x_n)$$

3. $\text{Ave} \approx f(x_1)\frac{\Delta x}{b-a} + \cdots + f(x_n)\frac{\Delta x}{b-a}$

$$\text{Average } \approx \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

Thus, we can define

$$\text{Average} = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

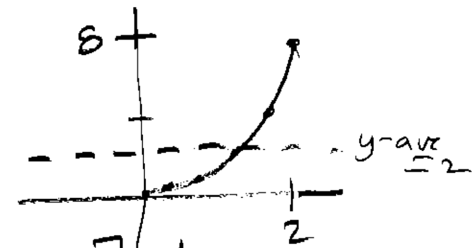
Which means the exact average y-value of y = f(x) over x = a to x = b is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Ex) $f(x) = x^3$, $x=0$ to $x=2$

$\Delta x = \frac{2-0}{5} = \frac{2}{5}$

$0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, 2$



$\left[\left(\frac{2}{5}\right)^3 + \left(\frac{4}{5}\right)^3 + \left(\frac{6}{5}\right)^3 + \left(\frac{8}{5}\right)^3 + (2)^3\right]\frac{1}{5}$

$\Delta x = \frac{2}{5} \Rightarrow \frac{\Delta x}{(2)} = \frac{1}{5}$ $\quad 2-0$

$\frac{1}{2}\underbrace{\left[\left(\frac{2}{5}\right)^3\Delta x + \left(\frac{4}{5}\right)^3\Delta x + \cdots + \left(\frac{8}{5}\right)^3\Delta x + (2)^3\Delta x\right]}$

$2-0$

$\frac{1}{b-a}\int_a^b f(x)dx$

$\frac{1}{2-0}\int_0^2 x^3 dx$

$\frac{1}{2}\left(\frac{1}{4}x^4\Big|_0^2\right)$

$\frac{1}{8}\left((2^4) - (0^4)\right) = \frac{16}{8} = \boxed{2}$

y-ave = 2, it occurs when $x = 2^{1/3}$